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# $(\alpha,\beta)-$Level Bipolar Smooth Fuzzy Soft Subgroup Structures

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**Abstract**. In this paper, we define a new kind of fuzzy set which bipolar smooth fuzzy soft set by combining the fuzzy star shaped set and bipolar smooth fuzzy soft set. By the basic definition of bipolar smooth fuzzy soft star shaped set, we discuss the relationships among these different types of star shapedness and obtain some properties. Finally, we investigate $(\alpha,\beta)-$level bipolar smooth fuzzy soft set and its properties.

**Keywords:** *Soft set, bipolar fuzzy set, bipolar smooth fuzzy soft set ,* $(\alpha,\beta)-$*level set, bipolar fuzzy soft pseudo-star shaped set, linear invertible transformation, quasi-convex star shaped set.*

## 1. Introduction

Zadeh [31] introduced the concept of fuzzy set as a new mathematical tool for dealing with uncertainties. There are several kinds of fuzzy set extensions in the fuzzy set theory,for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, rough fuzzy sets, soft fuzzy sets, vague sets, etc [19]. Bipolar-valued fuzzy set is another extension of fuzzy set whose membership degree range is extended from the interval [0,1] to the interval [-1,1]. The idea of bipolar valued fuzzy set was introduced by K.M.Lee [21, 23], as a generalization of the notion of fuzzy set. Since then, the theory of bipolar valued fuzzy sets has become a vigorous area of research in different disciplines such as algebraic structure, medical science, graph theory, decision making, machine theory and so on [2, 14, 15, 16, 20]. Convexity plays a most useful role in the theory and applications of fuzzy sets. In the basic and classical paper [31], Zadeh paid special attention to the investigation of the convex fuzzy sets. Following the seminal work of Zadeh on the definition of a convex fuzzy set, Ammar and Metz defined another type of convex fuzzy sets in [1]. From then on, Zadeh's convex fuzzy sets were called quasi-convex fuzzy sets in order to avoid misunderstanding. A lot of scholars have discussed various aspects of the theory and applications of quasi-convex fuzzy sets and convex fuzzy sets [6,7,13,14,22,24,26,27,30]. However, nature is not convex and, apart from possible applications, it is of independent interest to see how far the supposition of convexity can be weakened without losing too much structure. Star shaped sets are a fairly natural extension which is also an important issue in classical convex analysis [4,8,12,28]. Many remarkable theorems such as Helly's Theorem [33] and Krasnosel'skii's Theorem [18] relate to star shaped sets. In [6], Brown introduced the concept of star shaped fuzzy sets, and in [10] Diamond defined another type of star shaped fuzzy sets and established some of the basic properties of this family of fuzzy sets. To avoid misunderstanding, Brown's star shaped fuzzy sets will be called quasi-star shaped fuzzy sets in the sequel. Recently, the research of fuzzy star shaped (f.s.) set has been again attracting the deserving attention [5,9,21]. But with regards to Diamond's definition, there exists a small mistake which has been corrected in [10]. Motivated both by Diamond's research and by the importance of the concept of fuzzy convexity, we propose in this paper new and more general definition in the area of fuzzy star shapedness. Shadows of fuzzy set is another important concept in the classical paper [32]. In [31,32] Zadeh, and in [24] Liu obtained some interesting results on the shadows of convex fuzzy sets. Recently, some authors made further investigation about this subject [3,11]. Inspired by Liu's work [24], we will present fundamental discussion about shadows of star shaped fuzzy sets. In this paper, we define a new kind of fuzzy set which bipolar smooth fuzzy soft set by combining the fuzzy star shaped set and bipolar smooth fuzzy soft set. By the basic definition of bipolar smooth fuzzy soft star shaped set, we discuss the relationships among these different types of star shapedness and obtain some properties. Finally, we investigate level bipolar smooth fuzzy soft set and its properties.

## 2. Star shaped ness bipolar fuzzy soft set

*Definition 2.1*

Let $x, y \in R^n$, $\overline{xy} = \{ z / z = \alpha x + \beta y \}$ is a line segment, where $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$. A set S is simply said to be star shaped relative to a point $x \in R^n$, if $\overline{xy} \subseteq S$ for any point $y \in S$. The kernel of S is the set of all points $x \in S$ such that $\overline{xy} \subseteq S$ for any $y \in S$.

*Definition 2.2*

Let $R^n$ denote an universe of discourse. A bipolar fuzzy soft set $\overline{A}$ is an object having the form $\overline{A} = \left\{ ( x, \delta_{\overline{A}}{}^{P}(x), \delta_{\overline{A}}{}^{N}(x)) / x \in R^n \right\}$ where $\delta_{\overline{A}}{}^{P} : R^n \to [0,1]$ and $\delta_{\overline{A}}{}^{N} : R^n \to [-1,0]$ satisfy $-1 \leq \delta_{A}{}^{P} + \delta_{A}{}^{N} \leq 1$ for all $x \in R^n$, $\delta_{\overline{A}}{}^{P}$ and $\delta_{\overline{A}}{}^{N}$ are called the degree of positive membership function and degree of negative membership function of the element $x$ to $\overline{A}$ respectively.

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Let $F(R^n)$ be the classes of normal BFS-sets of $R^n$, (ie) $\left\{x \in R^n / \delta_{\bar{A}}{}^{P}(x) / \delta_{\bar{A}}{}^{P}(x) = 1 \text{ and } \delta_{\bar{A}}{}^{N}(x) = -1\right\}$ is non empty.

*Example 2.1*

Let $\bar{A} = \left\{(x, \delta_{\bar{A}}{}^{P}(x), \delta_{\bar{A}}{}^{N}(x)) / x \in R\right\}$

where

$$\delta_{\bar{A}}{}^{P}(x) = \begin{cases} x+1 & x \in [-1,0] \\ -x+1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

and $\delta_{\bar{A}}{}^{N}(x) = \begin{cases} -x & x \in [-1,0] \\ x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$.

Then $\bar{A} \in F(R)$

*Definition 2.3*

A BFS-set $\bar{A} \in F(R^n)$ is called quasi-convex if

$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \inf\left\{\delta_{\bar{A}}{}^{P}(x), \delta_{\bar{A}}{}^{P}(y)\right\}$ and

$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \sup\left\{\delta_{\bar{A}}{}^{N}(x), \delta_{\bar{A}}{}^{N}(y)\right\}$

for all $x, y \in R^n$, $\lambda \in [-1,1]$.

*Definition 2.4*

A BFS- set $\bar{A} \in F(R^n)$ is said to be bipolar fuzzy soft star shaped set relative to $y \in R^n$ if $\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \delta_{\bar{A}}{}^{P}(x)$ and $\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \delta_{\bar{A}}{}^{N}(x)$ for all $x \in R^n$, $\lambda \in [-1,1]$.

*Proposition 2.1*

Let $\bar{A} \in F(R^n)$ is bipolar fuzzy soft set relative to $y \in R^n$. Then

$$\delta_{\bar{A}}{}^{P}(x) = \sup_{x \in R^n}\left\{\delta_{\bar{A}}{}^{P}(x)\right\} = 1$$

$$\delta_{\bar{A}}{}^{N}(x) = \inf_{x \in R^n}\left\{\delta_{\bar{A}}{}^{N}(x)\right\} = -1$$

*Proof:*

Let $\bar{A}$ be bipolar fuzzy soft relative to y. Then for all $x \in R^n$

$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \delta_{\bar{A}}{}^{P}(x)$ and

$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \delta_{\bar{A}}{}^{N}(x)$ are true for $-1 \leq \lambda \leq 1$.

Thus, only take $\lambda = 0$, it can be found that $\delta_{\bar{A}}{}^{P}(y) \geq \delta_{\bar{A}}{}^{P}(x)$ and $\delta_{\bar{A}}{}^{N}(y) \leq \delta_{\bar{A}}{}^{N}(x)$ are true for all $x \in R^n$.

Hence $\delta_{\bar{A}}{}^{P}(x) = \sup_{x \in R^n}\left\{\delta_{\bar{A}}{}^{P}(x)\right\} = 1$

$\delta_{\bar{A}}{}^{N}(x) = \inf_{x \in R^n}\left\{\delta_{\bar{A}}{}^{N}(x)\right\} = -1$.

*Example 2.2*

A bipolar fuzzy soft set $\bar{A} \in F(R^n)$ with

$$\delta_{\bar{A}}{}^{P}(x) = \begin{cases} e^{x} & x \in (-\infty, 0] \\ e^{-x} & x \in (0, \infty) \end{cases}$$

$$\delta_{\bar{A}}{}^{N}(x) = \begin{cases} 1-e^{x} & x \in (-\infty, 0] \\ 1-e^{-x} & x \in (0, \infty) \end{cases}$$

is bipolar fuzzy soft set relative to $y = 0$.

*Proposition 2.2*

A bipolar fuzzy soft set $\bar{A} \in F(R^n)$ is relative to $y \in R^n$ if and only if its level sets are star shaped relative to y.

*Proof:*

Suppose $\delta_{\bar{A}}^{[\alpha,\beta]}$ is star shaped relative to $y \in R^n$ for all $\alpha, \beta \in [-1,1]$.

For $x \in R^n$, let $\alpha = \delta_{\bar{A}}{}^{P}(x)$, $\beta = \delta_{\bar{A}}{}^{N}(x)$ then $\overline{xy} \in \delta_{\bar{A}}^{[\alpha,\beta]}$ that is for any $\lambda \in [-1,1]$

$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \alpha = \delta_{\bar{A}}{}^{P}(x)$ and

$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \beta = \delta_{\bar{A}}{}^{N}(x)$

Conversely, if for all $x \in R^n$, $\lambda \in [-1,1]$.

$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \delta_{\bar{A}}{}^{P}(x)$ and

$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \delta_{\bar{A}}{}^{N}(x)$ hold.

Since $\delta_{\bar{A}}^{[\alpha,\beta]} \neq \phi$, there exists $x \in \delta_{\bar{A}}^{[\alpha,\beta]}$, which means $\delta_{\bar{A}}{}^{P}(x) \geq \alpha$ and $\delta_{\bar{A}}{}^{N}(x) \leq \beta$.

Hence $\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \delta_{\bar{A}}{}^{P}(x) \geq \alpha$ and

$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \delta_{\bar{A}}{}^{N}(x) \leq \beta$ for any $\lambda \in [-1,1]$.

So $\overline{xy} \in \delta_{\bar{A}}^{[\alpha,\beta]}$. Then $\delta_{\bar{A}}^{[\alpha,\beta]}$ is star shaped relative to y.

*Definition 2.5*

A bipolar fuzzy soft set (BFSS) $\bar{A} \in F(R^n)$ is said to be bipolar generalised fuzzy soft set (BGFSS) if all level sets are star shaped sets in $R^n$.

*Definition 2.6*

A BFS-set $\bar{A} \in F(R^n)$ is said to be bipolar fuzzy soft quasi set (BFSQS) relative to $x \in R^n$, if for all $x \in R^n$, $\lambda \in [-1,1]$, the following hold

$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \inf\left\{\delta_{\bar{A}}{}^{P}(x), \delta_{\bar{A}}{}^{P}(y)\right\}$,

$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \sup\left\{\delta_{\bar{A}}{}^{N}(x), \delta_{\bar{A}}{}^{N}(y)\right\}$

*Definition 2.7*

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A BFS-set $\bar{A} \in F(R^n)$ is said to be bipolar fuzzy soft pseudo set (BFSPS) relative to $x \in R^n$ , $\lambda \in [-1,1]$ , the following are true

$$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \lambda\,\delta_{\bar{A}}{}^{P}(x)+(1-\lambda)\,\delta_{\bar{A}}{}^{P}(y),$$

$$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \lambda\,\delta_{\bar{A}}{}^{N}(x)+(1-\lambda)\,\delta_{\bar{A}}{}^{N}(y)$$

*Definition 2.8*

A bipolar fuzzy soft hypo graph of $\bar{A}$ denoted by for $f.hypo(\bar{A})$, is defined as

$$f.hypo(\bar{A}) = f.hypo(\delta_{\bar{A}}{}^{P}) \cup f.hypo(\delta_{\bar{A}}{}^{N}) \text{ where}$$

$$f.hypo(\delta_{\bar{A}}{}^{P}) = \{(x,t)/x \in R, t \in [-1, \delta_{\bar{A}}{}^{P}(x)]\}$$

$$f.hypo(\delta_{\bar{A}}{}^{N}) = \{(x,s)/x \in R, s \in [\delta_{\bar{A}}{}^{N}(x), 1]\}.$$

*Theorem 2.1*

Let $\bar{A} \in F(R^n)$ is BFSP-set relative to $y \in R^n$ . Then it is BFSQ-set relative to y.

*Proof:*

Since for all $x \in R^n$ , $\lambda \in [-1,1]$ , the following hold,

$$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \lambda\,\delta_{\bar{A}}{}^{P}(x)+(1-\lambda)\,\delta_{\bar{A}}{}^{P}(y) \geq \inf\{\delta_{\bar{A}}{}^{P}(x), \delta_{\bar{A}}{}^{P}(y)\}$$

$$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \lambda\,\delta_{\bar{A}}{}^{N}(x)+(1-\lambda)\,\delta_{\bar{A}}{}^{N}(y) \leq \sup\{\delta_{\bar{A}}{}^{N}(x), \delta_{\bar{A}}{}^{N}(y)\}$$

.

Thus $\bar{A}$ is BFSQ-set relative to y.

*Note 1:*

The converse statements do not hold in general as shown in the following example.

*Example 2.3*

A bipolar fuzzy soft set with the positive membership function

$$\delta_{\bar{A}}{}^{P}(x) = \begin{cases} 2+x & x \in [-2,-1] \\ x^2 & x \in [-1,1] \\ 2-x & x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

and the negative membership function

$$\delta_{\bar{A}}{}^{N}(x) = \begin{cases} -x-1 & x \in [-2,-1] \\ 1-x^2 & x \in [-1,1] \\ x-1 & x \in [1,2] \\ 1 & \text{otherwise} \end{cases}$$

is BFSQ-set relative to $y=0$. But it is not BFSP-set relative to $y=0$.

*Theorem 2.2*

Let $\bar{A} \in F(R^n)$ be BFSQ-set relative to $y \in R^n$ .Then

$$\delta_{\bar{A}}{}^{P}(y) = \sup_{x \in R^n}\{\delta_{\bar{A}}{}^{P}(x)\} = 1$$

$$\delta_{\bar{A}}{}^{N}(y) = \inf_{x \in R^n}\{\delta_{\bar{A}}{}^{N}(x)\} = -1$$

if and only if $\bar{A} \in F(R^n)$ is BFS-set relative to y.

*Proof:*

*Necessary Condition:*

Since $\bar{A} \in F(R^n)$ is BFSQ-set relative to $y \in R^n$ ,

$$\delta_{\bar{A}}{}^{P}(y) = \sup_{x \in R^n}\{\delta_{\bar{A}}{}^{P}(x)\} = 1 \text{ and}$$

$$\delta_{\bar{A}}{}^{N}(y) = \inf_{x \in R^n}\{\delta_{\bar{A}}{}^{N}(x)\} = -1$$

then for all $x \in R^n$ , $\lambda \in [-1,1]$.

We have

$$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \inf\{\delta_{\bar{A}}{}^{P}(x), \delta_{\bar{A}}{}^{P}(y)\} = \delta_{\bar{A}}{}^{P}(x)$$

$$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \sup\{\delta_{\bar{A}}{}^{N}(x), \delta_{\bar{A}}{}^{N}(y)\} = \delta_{\bar{A}}{}^{N}(x).$$

Hence $\bar{A}$ is BFS-set relative to y.

*Sufficient condition:*

Since $\bar{A}$ is BFS-set relative to y, which means for all $x \in R^n$ , $\lambda \in [-1,1]$ ,

$$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \delta_{\bar{A}}{}^{P}(x) \text{ and}$$

$$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \delta_{\bar{A}}{}^{N}(x)$$

Take $\lambda = 0$ , we get $\delta_{\bar{A}}{}^{P}(y) \geq \delta_{\bar{A}}{}^{P}(x)$ and $\delta_{\bar{A}}{}^{N}(y) \leq \delta_{\bar{A}}{}^{N}(x)$ for all $x \in R^n$ .

Thus,

$$\delta_{\bar{A}}{}^{P}(\lambda(x-y)+y) \geq \delta_{\bar{A}}{}^{P}(x) \geq \inf\{\delta_{\bar{A}}{}^{P}(x), \delta_{\bar{A}}{}^{P}(y)\}$$

$$\delta_{\bar{A}}{}^{N}(\lambda(x-y)+y) \leq \delta_{\bar{A}}{}^{N}(x) \leq \sup\{\delta_{\bar{A}}{}^{N}(x), \delta_{\bar{A}}{}^{N}(y)\}$$

Hence $\bar{A}$ is BFSQ-set relative to y.

$$\delta_{\bar{A}}{}^{P}(y) = \sup_{x \in R^n}\{\delta_{\bar{A}}{}^{P}(x)\} = 1$$

$$\delta_{\bar{A}}{}^{N}(y) = \inf_{x \in R^n}\{\delta_{\bar{A}}{}^{N}(x)\} = -1$$

*Theorem 2.3*

Let $\bar{A} \in F(R^n)$ is BFSP-set relative to $y \in R^n$ . If

$$\delta_{\bar{A}}{}^{P}(y) = \sup_{x \in R^n}\{\delta_{\bar{A}}{}^{P}(x)\} = 1 \text{ and}$$

$$\delta_{\bar{A}}{}^{N}(y) = \inf_{x \in R^n}\{\delta_{\bar{A}}{}^{N}(x)\} = -1.$$

Then it is BFS-set relative to y.

*Proof:*

It follows from theorem (2.1) and theorem (2.2).

*Note 2*

The converse of the above theorem does not hold in general as shown in the following example.

*Example 2. 4*

A bipolar fuzzy soft set $\bar{A} \in F(R^n)$ with

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$$\delta_{\bar{A}}^{\ P}(x) = \begin{cases} e^{x} & x \in (-\infty, 0) \\ e^{-x} & x \in [0, \infty) \end{cases}$$

$$\delta_{\bar{A}}^{\ N}(x) = \begin{cases} 1-e^{x} & x \in (-\infty, 0] \\ 1-e^{-x} & x \in (0, \infty) \end{cases}$$

is bipolar fuzzy soft set relative to $y = 0$. But it is not BFSP-set relative to $y = 0$.

*Theorem 2.4*

A bipolar fuzzy soft set (BFSS-set) $\bar{A} \in F(R^n)$ is bipolar fuzzy soft star shaped relative to $y \in R^n$ iff for all $x \in R^n$, $\lambda \in [-1,1]$, the following hold,

$\delta_{\bar{A}}^{\ P}(\lambda x + y) \geq \delta_{\bar{A}}^{\ P}(x+y)$ and

$\delta_{\bar{A}}^{\ N}(\lambda x + y) \leq \delta_{\bar{A}}^{\ N}(x+y)$.

*Proof*

Suppose $\bar{A}$ is bipolar fuzzy soft star shaped relative to y , that is for all $x \in R^n$, $\lambda \in [-1,1]$,

$$\left.\begin{array}{l} \delta_{\bar{A}}^{\ P}(\lambda(x-y)+y) \geq \delta_{\bar{A}}^{\ P}(x) \text{ and} \\ \delta_{\bar{A}}^{\ N}(\lambda(x-y)+y) \leq \delta_{\bar{A}}^{\ N}(x) \end{array}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

Replacing $x$ by $x+y$ in the above equation (1), we can get

$\delta_{\bar{A}}^{\ P}(\lambda(x+y-y)+y) \geq \delta_{\bar{A}}^{\ P}(x+y)$

$\Rightarrow \delta_{\bar{A}}^{\ P}(\lambda x+y) \geq \delta_{\bar{A}}^{\ P}(x+y)$ is proved and

$\delta_{\bar{A}}^{\ N}(\lambda(x+y-y)+y) \leq \delta_{\bar{A}}^{\ N}(x+y)$

$\Rightarrow \delta_{\bar{A}}^{\ N}(\lambda x+y) \leq \delta_{\bar{A}}^{\ N}(x+y)$ is proved.

Similarly, we can get the converse part.

*Theorem 2.5*

A bipolar fuzzy soft set $\bar{A} \in F(R^n)$ is BFSQ-set relative to $y \in R^n$ iff $\delta_{\bar{A}}^{[\alpha,\beta]}$ is star shaped set relative to y for $\alpha \in [-1, \delta_{\bar{A}}^{\ P}(y)]$, $\beta \in [\delta_{\bar{A}}^{\ N}(y), 1]$.

*Proof:*

*Necessary condition:*

Suppose $\bar{A}$ is BFSQ-set relative to y, that is for all $x \in R^n$, $\lambda \in [-1,1]$ ,

$$\delta_{\bar{A}}^{\ P}(\lambda(x-y)+y) \geq \inf\left\{\delta_{\bar{A}}^{\ P}(x), \delta_{\bar{A}}^{\ P}(y)\right\}$$

$$\delta_{\bar{A}}^{\ N}(\lambda(x-y)+y) \leq \sup\left\{\delta_{\bar{A}}^{\ N}(x), \delta_{\bar{A}}^{\ N}(y)\right\}.$$

For any $\alpha \in [-1, \delta_{\bar{A}}^{\ P}(y)]$ , $\beta \in [\delta_{\bar{A}}^{\ N}(y), 1]$ , if $x \in \delta_{\bar{A}}^{[\alpha,\beta]}$, then we have that $x, y \in \delta_{\bar{A}}^{[\alpha,\beta]}$.

From the above inequality,

We get that $\delta_{\bar{A}}^{\ P}(\lambda(x-y)+y) \geq \alpha$ and $\delta_{\bar{A}}^{\ N}(\lambda(x-y)+y) \leq \beta$. So $\overline{xy} \in \delta_{\bar{A}}^{[\alpha,\beta]}$.

*Sufficient condition:*

*Case (i)*

For $x \in R^n$, $\lambda \in [-1,1]$, if $\delta_{\bar{A}}^{\ P}(x) \geq \delta_{\bar{A}}^{\ P}(y)$, then let $\alpha = \delta_{\bar{A}}^{\ P}(y)$. Accordingly we have $\overline{xy} \in \delta_{\bar{A}}^{[\alpha,\beta]}$, that is $\delta_{\bar{A}}^{\ P}(\lambda(x-y)+y) \geq \inf\left\{\delta_{\bar{A}}^{\ P}(x), \delta_{\bar{A}}^{\ P}(y)\right\}$.

*Case (ii)*

If $\delta_{\bar{A}}^{\ P}(x) \leq \delta_{\bar{A}}^{\ P}(y)$ , then let $\alpha = \delta_{\bar{A}}^{\ P}(x)$ . Accordingly we have $\overline{xy} \in \delta_{\bar{A}}^{[\alpha,\beta]}$ , that is $\delta_{\bar{A}}^{\ P}(\lambda(x-y)+y) \geq \inf\left\{\delta_{\bar{A}}^{\ P}(x), \delta_{\bar{A}}^{\ P}(y)\right\}$.

*Case (iii)*

If $\delta_{\bar{A}}^{\ N}(x) \leq \delta_{\bar{A}}^{\ N}(y)$ , then let $\beta = \delta_{\bar{A}}^{\ N}(y)$ . Accordingly we have $\overline{xy} \in \delta_{\bar{A}}^{[\alpha,\beta]}$ , that is $\delta_{\bar{A}}^{\ N}(\lambda(x-y)+y) \leq \sup\left\{\delta_{\bar{A}}^{\ N}(x), \delta_{\bar{A}}^{\ N}(y)\right\}$.

*Case (iv)*

If $\delta_{\bar{A}}^{\ N}(x) \geq \delta_{\bar{A}}^{\ N}(y)$ , then let $\beta = \delta_{\bar{A}}^{\ N}(x)$ . Accordingly we have $\overline{xy} \in \delta_{\bar{A}}^{[\alpha,\beta]}$ , that is $\delta_{\bar{A}}^{\ N}(\lambda(x-y)+y) \leq \sup\left\{\delta_{\bar{A}}^{\ N}(x), \delta_{\bar{A}}^{\ N}(y)\right\}$.

Thus $\bar{A}$ is BFSQ-set relative to $y \in R^n$.

*Theorem 2.6*

A bipolar fuzzy soft set $\bar{A} \in F(R^n)$ is BFSP-set relative to y iff $f.hypo\left(\delta_{\bar{A}}^{\ P}\right)$ is star shaped relative to $(y, \delta_{\bar{A}}^{\ P}(y))$ and $f.hypo\left(\delta_{\bar{A}}^{\ N}\right)$ is star shaped relative to $(y, \delta_{\bar{A}}^{\ N}(y))$.

*Proof:*

*Necessary condition:*

If $\bar{A}$ is BFSP-set relative to y, $(x,t) \in f.hypo\left(\delta_{\bar{A}}^{\ P}\right)$ and $(x,s) \in f.hypo\left(\delta_{\bar{A}}^{\ N}\right)$.

Since $\bar{A}$ is BFSP-set relative to y.

For any $\lambda \in [-1,1]$, we have

$$\delta_{\bar{A}}^{\ P}(\lambda(x-y)+y) \geq \lambda\delta_{\bar{A}}^{\ P}(x) + (1-\lambda)\delta_{\bar{A}}^{\ P}(y)$$
$$\geq \lambda t + (1-\lambda)\delta_{\bar{A}}^{\ P}(y)$$

$$\delta_{\bar{A}}^{\ N}(\lambda(x-y)+y) \leq \lambda\delta_{\bar{A}}^{\ N}(x) + (1-\lambda)\delta_{\bar{A}}^{\ N}(y)$$
$$\leq \lambda s + (1-\lambda)\delta_{\bar{A}}^{\ N}(y)$$

Thus, we have

$\lambda(x,t) + (1-\lambda)(y, \delta_{\bar{A}}^{\ P}(y)) \in f.hypo\left(\delta_{\bar{A}}^{\ P}\right)$

$\lambda(x,s) + (1-\lambda)(y, \delta_{\bar{A}}^{\ N}(y)) \in f.hypo\left(\delta_{\bar{A}}^{\ N}\right)$

Hence $f.hypo\left(\delta_{\bar{A}}^{\ P}\right)$ is star shaped relative to $(y, \delta_{\bar{A}}^{\ P}(y))$ and $f.hypo\left(\delta_{\bar{A}}^{\ N}\right)$ is star shaped relative to $(y, \delta_{\bar{A}}^{\ N}(y))$.

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*Sufficient condition:*

Assume that $(x,\delta_{\overline{A}}{}^{P}(x)) \in f.hypo\left(\delta_{\overline{A}}{}^{P}\right)$ and $(x,\delta_{\overline{A}}{}^{N}(x)) \in f.hypo\left(\delta_{\overline{A}}{}^{N}\right)$. By the star shapedness of $f.hypo\left(\delta_{\overline{A}}{}^{P}\right)$ and $f.hypo\left(\delta_{\overline{A}}{}^{N}\right)$, we can have

$$\lambda x+(1-\lambda)y\ ,\ \lambda\delta_{\overline{A}}{}^{P}(x)+(1-\lambda)\delta_{\overline{A}}{}^{P}(y) \in f.hypo\left(\delta_{\overline{A}}{}^{P}\right)$$

$$\lambda x+(1-\lambda)y\ ,\ \lambda\delta_{\overline{A}}{}^{N}(x)+(1-\lambda)\delta_{\overline{A}}{}^{N}(y) \in f.hypo\left(\delta_{\overline{A}}{}^{N}\right)$$

For any $\lambda\in[-1,1]$. Thus $\overline{A}$ is BFSP-set relative to y.

*Definition 2.9*

A path in a set S in $R^n$ is a continuous mapping $f:[-1,1]\to S$. A set S is said to be path connected, if there exist a path f such that $f(-1)=x$ and $f(1)=y$ for all $x,y\in S$.

*Note 3*

a) A set BFS-set $\overline{A}$ is said to be path connected if its level sets are path connected.

b) Since a star shaped crisp set is path connected.

*Proposition 2.3*

If $\overline{A}\in F(R^n)$ is BFS-set relative to $y\in R^n$, then $\overline{A}$ is a path connected bipolar fuzzy soft set.

*Proof:*

It follows from definitions (2.4) and (2.5)

*Proposition 2.4*

If $\overline{A}\in F(R^n)$ is bipolar fuzzy soft quasi convex set, then it is BFSGS. Furthermore, if $\overline{A}\in F(R^n)$ is BFSGS. Then $\overline{A}$ is a BFSQC-set.

*Proof:*

If $\overline{A}$ is a BFSQC-set, that for all $x,y\in R^n$, $\lambda\in[-1,1]$, we have

$$\delta_{\overline{A}}{}^{P}(\lambda(x-y)+y) \geq \inf\left\{\delta_{\overline{A}}{}^{P}(x),\delta_{\overline{A}}{}^{P}(y)\right\}$$

$$\delta_{\overline{A}}{}^{N}(\lambda(x-y)+y) \leq \sup\left\{\delta_{\overline{A}}{}^{N}(x),\delta_{\overline{A}}{}^{N}(y)\right\}$$

then for all $x,y\in R^n$, the following hold,

$$\delta_{\overline{A}}{}^{P}(\lambda(x-y)+y) \geq \inf\left\{\delta_{\overline{A}}{}^{P}(x),\delta_{\overline{A}}{}^{P}(y)\right\} \geq \inf\{\alpha,\alpha\} \geq \alpha$$

$$\delta_{\overline{A}}{}^{N}(\lambda(x-y)+y) \leq \sup\left\{\delta_{\overline{A}}{}^{N}(x),\delta_{\overline{A}}{}^{N}(y)\right\} \leq \sup\{\beta,\beta\} \leq \beta$$

.

So $\overline{xy}\in\delta_{\overline{A}}{}^{[\alpha,\beta]}$. In other words $\overline{A}$ is a BFSGS-set.

Additionally if $\overline{A}\in F(R^n)$ is BFSGS-set , then $\delta_{\overline{A}}{}^{[\alpha,\beta]}$ is star shaped. Thus they are path convex. So we have that $\overline{A}$ is BFS-set relative to y and is a BFSQC-set.

*Proposition 2.5*

If $\overline{A}\in F(R^n)$ is bipolar fuzzy soft set and the point $y\in R^n$ satisfies that $\delta_{\overline{A}}{}^{P}(y)=\inf_{x\in R}\left\{\delta_{\overline{A}}{}^{P}(x)\right\}$ and $\delta_{\overline{A}}{}^{N}(y)=\sup_{x\in R}\left\{\delta_{\overline{A}}{}^{N}(x)\right\}$. Then $\overline{A}$ is BFSQS-set relative to y, that is $y\in q-\ker(\overline{A})$.

*Proof:*

By applying definition of BFSQS-set, we have $\delta_{\overline{A}}{}^{P}(y)=\inf_{x\in R}\left\{\delta_{\overline{A}}{}^{P}(x)\right\}$ and $\delta_{\overline{A}}{}^{N}(y)=\sup_{x\in R}\left\{\delta_{\overline{A}}{}^{N}(x)\right\}$. So the given statement is true.

*Proposition 2.6*

If $\overline{A}_1,\overline{A}_2\in F(R^n)$ are BFSQS (respectively BFSPS) relative to $y\in R^n$, then $\overline{A}_1\cap\overline{A}_2$ is BFSQS (respectively BFSPS) relative to y.

*Proof:*

Because that $\overline{A}_1,\overline{A}_2\in F(R^n)$ are BFSQS relative to $y\in R^n$, for all $x\in R^n$, $\lambda\in[-1,1]$, we have

$$\delta_{\overline{A}_i}{}^{P}(\lambda x+(1-\lambda)y) \geq \inf\left\{\delta_{\overline{A}_i}{}^{P}(x),\delta_{\overline{A}_i}{}^{P}(y)\right\}$$

$$\delta_{\overline{A}_i}{}^{N}(\lambda x+(1-\lambda)y) \leq \sup\left\{\delta_{\overline{A}_i}{}^{N}(x),\delta_{\overline{A}_i}{}^{N}(y)\right\}$$

$i=1,2$

So,

$$\begin{aligned}(\delta_{\overline{A}_1}{}^{P}\cap\delta_{\overline{A}_2}{}^{P})(\lambda x+(1-\lambda)y) &= \inf\left\{\delta_{\overline{A}_i}{}^{P}(\lambda x+(1-\lambda)y),\ i=1,2\right\}\\ &\geq \inf\left\{\sup\left\{\delta_{\overline{A}_1}{}^{P}(x),\delta_{\overline{A}_1}{}^{P}(y)\right\},\sup\left\{\delta_{\overline{A}_2}{}^{P}(x),\delta_{\overline{A}_2}{}^{P}(y)\right\}\right\}\\ &= \inf\left\{(\delta_{\overline{A}_1}{}^{P}\cap\delta_{\overline{A}_2}{}^{P})(x),(\delta_{\overline{A}_1}{}^{P}\cap\delta_{\overline{A}_2}{}^{P})(y)\right\}\end{aligned}$$

and

$$\begin{aligned}(\delta_{\overline{A}_1}{}^{N}\cup\delta_{\overline{A}_2}{}^{N})(\lambda x+(1-\lambda)y) &= \sup\left\{\delta_{\overline{A}_i}{}^{N}(\lambda x+(1-\lambda)y),\ i=1,2\right\}\\ &\leq \sup\left\{\sup\left\{\delta_{\overline{A}_1}{}^{N}(x),\delta_{\overline{A}_1}{}^{N}(y)\right\},\sup\left\{\delta_{\overline{A}_2}{}^{N}(x),\delta_{\overline{A}_2}{}^{N}(y)\right\}\right\}\\ &= \sup\left\{(\delta_{\overline{A}_1}{}^{N}\cup\delta_{\overline{A}_2}{}^{N})(x),(\delta_{\overline{A}_1}{}^{N}\cup\delta_{\overline{A}_2}{}^{N})(y)\right\}\end{aligned}$$

Hence $\overline{A}_1\cap\overline{A}_2$ is BFSQS relative to y.

If $\overline{A}_1,\overline{A}_2\in F(R^n)$ are BFSPS relative to $y\in R^n$, then for all $x\in R^n$, $\lambda\in[-1,1]$, we have

$$\delta_{\overline{A}_i}{}^{P}(\lambda(x-y)+y) \geq \lambda\delta_{\overline{A}_i}{}^{P}(x)+(1-\lambda)\delta_{\overline{A}_i}{}^{P}(y)$$

$$\delta_{\overline{A}_i}{}^{N}(\lambda(x-y)+y) \leq \lambda\delta_{\overline{A}_i}{}^{N}(x)+(1-\lambda)\delta_{\overline{A}_i}{}^{N}(y) \qquad ,\ i=1,2$$

So,

$$\begin{aligned}(\delta_{\overline{A}_1}{}^{P}\cap\delta_{\overline{A}_2}{}^{P})(\lambda x+(1-\lambda)y) &= \inf\left\{\delta_{\overline{A}_i}{}^{P}(\lambda x+(1-\lambda)y),\ i=1,2\right\}\\ &\geq \inf\left\{\lambda\delta_{\overline{A}_i}{}^{P}(x)+(1-\lambda)\delta_{\overline{A}_i}{}^{P}(\right.\\ &\geq \lambda\ \inf\left\{\delta_{\overline{A}_1}{}^{P}(x),\ \delta_{\overline{A}_2}{}^{P}(x)\right\}+(\\ &= \lambda\ (\delta_{\overline{A}_1}{}^{P}\cap\delta_{\overline{A}_2}{}^{P})(x)+(1-\lambda)\end{aligned}$$

and

$$\begin{aligned}(\delta_{\overline{A}_1}{}^{N}\cup\delta_{\overline{A}_2}{}^{N})(\lambda x+(1-\lambda)y) &= \sup\left\{\delta_{\overline{A}_i}{}^{N}(\lambda x+(1-\lambda)y),\ i\right.\\ &\leq \sup\left\{\lambda\delta_{\overline{A}_i}{}^{N}(x)+(1-\lambda)\delta_{\overline{A}_i}{}^{N}(\right.\end{aligned}$$

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$$\leq \lambda \sup\left\{\delta_{\bar{A}_1}{}^{N}(x),\ \delta_{\bar{A}_2}{}^{N}(x)\right\} + (1-\lambda)\sup\left\{\delta_{\bar{A}_1}{}^{N}(y),\ \delta_{\bar{A}_2}{}^{N}(y)\right\}$$
$$= \lambda\ (\delta_{\bar{A}_1}{}^{N} \cup \delta_{\bar{A}_2}{}^{N})(x) + (1-\lambda)(\delta_{\bar{A}_1}{}^{N} \cup \delta_{\bar{A}_2}{}^{N})(y)$$

Hence $\bar{A}_1 \cap \bar{A}_2$ is BFSPS relative to y.

*Proposition 2.7*

If $\bar{A} \in F(R^n)$ is BFS (respectively BFSPS, BFSQS) relative to y and $x_0 \in R^n$ . Then $x_0 + \bar{A}$ is BFS-set (respectively BFSPS, BFSQS) relative to $x_0 + y$.

*Proof:*

In this proposition, we only give the proof for the case of bipolar fuzzy soft star shapedness. Similarly, the others can be proved.

For any $x_0 \in R^n$ , since $\bar{A} \in F(R^n)$ is BFS-set relative to y.
We have

$$\begin{aligned}(x_0 + \delta_{\bar{A}}{}^{P})(\lambda(x-y-x_0)+y+x_0) &= \delta_{\bar{A}}{}^{P}(\lambda(x-y-x_0)+y) \\ &\geq \delta_{\bar{A}}{}^{P}(x-x_0) \\ &= (x_0 + \delta_{\bar{A}}{}^{P})(x)\end{aligned}$$

and

$$\begin{aligned}(x_0 + \delta_{\bar{A}}{}^{N})(\lambda(x-y-x_0)+y+x_0) &= \delta_{\bar{A}}{}^{N}(\lambda(x-y-x_0)+y) \\ &\leq \delta_{\bar{A}}{}^{N}(x-x_0) \\ &= (x_0 + \delta_{\bar{A}}{}^{N})(x)\end{aligned}$$

So, $x_0 + \bar{A}$ is bipolar fuzzy soft set relative to $x_0 + y$.

*Definition 2.10*

Let T be a linear invertible transformation on $R^n$ , $\bar{A} \in F(R^n)$ . Then by the Extension principle we have $(T(\bar{A}))(x) = \bar{A}(T^{-1}(x))$.

*Proposition 2.8*

If $\bar{A} \in F(R^n)$ is bipolar fuzzy soft star shaped set (respectively BFSPS, BFSQS) relative to y and T is a linear invertible transformation on $R^n$. Then $T(\bar{A})$ is bipolar fuzzy soft star shaped set (respectively BFSPS , BFSQS) relative to $T(y)$.

*Proof:*

We only give the proof for the case of BFSQS. Similarly, the others can be proved.
For any $x \in R^n$, since $\bar{A} \in F(R^n)$ is BFSQS relative to y.
We have

$$\begin{aligned}(T(\bar{A}))(\lambda x + (1-\lambda)T(y)) &= \bar{A}(\lambda T^{-1}(x) + (1-\lambda)y) \\ &\geq \inf\left\{\bar{A}(T^{-1}(x), \bar{A}(y)\right\} \\ &= \inf\left\{T(\bar{A}(x)), T(\bar{A})(T(y))\right\}\end{aligned}$$

Hence $T(\bar{A})$ is BFSQS relative to $T(y)$.

**3. $(\alpha,\beta)-$Level bipolar smooth fuzzy soft set (BSFSS)**

*Definition 3.1*

Let $\delta_A$ be a BSFSS over U. Then $(\alpha,\beta)-$level of BSFSS $\delta_A$, denoted by $\delta_A{}^{(\alpha,\beta)}$ , is defined as follows
$\delta_A{}^{(\alpha,\beta)} = \left\{x \in A / \delta_A{}^{P}(x) \geq \alpha \text{ and } \delta_A{}^{N}(x) \leq \beta\right\}$ for $\alpha \cap \beta = \phi$.
Note that if $\alpha = \phi$ or $\beta = U$ , then $\delta_A{}^{(\phi,U)} = \left\{x \in A / \delta_A{}^{P}(x) \neq \phi \text{ and } \delta_A{}^{N}(x) = U\right\}$ is called Support of $\delta_A$, and denoted by Supp($\delta_A$).

*Example 5*

Let $U = \{u_1, u_2, u_3, \cdots, u_7\}$ be an initial universe and $E = \{e_1, e_2, e_3 \cdots, e_5\}$ be a parameter set. If we define BSFS as follows

| | |
|---|---|
| $\delta_A{}^{P}(e_1) = \phi$ | $\delta_A{}^{N}(e_1) = \{u_1, u_2, u_3\}$ |
| $\delta_A{}^{P}(e_2) = \{u_2, u_3, u_4, u_5, u_6\}$ | $\delta_A{}^{N}(e_2) = \{u_1\}$ |
| $\delta_A{}^{P}(e_3) = \{u_5, u_6, u_7\}$ | $\delta_A{}^{N}(e_3) = \{u_2, u_4\}$ |
| $\delta_A{}^{P}(e_4) = \phi$ | $\delta_A{}^{N}(e_4) = U$ |
| $\delta_A{}^{P}(e_5) = \{u_3, u_5, u_7\}$ | $\delta_A{}^{N}(e_5) = \{u_2, u_4\}$ |

Let $\alpha = \{u_5, u_7\}$ and $\beta = \{u_2, u_4, u_6\}$ . Then $\delta_A{}^{(\alpha,\beta)} = \{e_3, e_5\}$.

*Proposition 3.1*

Let $\delta_A$ and $\delta_B$ be two BSFS over U. $A, B \subseteq E$. Then following assertions hold:

1. $\delta_A \leq \delta_B \Rightarrow \delta_A{}^{(\alpha,\beta)} \subseteq \delta_B{}^{(\alpha,\beta)}$, for all $\alpha, \beta \subseteq U$ such that $\alpha \cap \beta = \phi$.
2. If $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$, then $\delta_A{}^{(\alpha_2,\beta_2)} \subseteq \delta_A{}^{(\alpha_1,\beta_1)}$ for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \subseteq U$ such that $\alpha_1 \cap \beta_1 = \phi$ or $\alpha_2 \cap \beta_2 = \phi$.
3. $\delta_A = \delta_B \Rightarrow \delta_A{}^{(\alpha,\beta)} = \delta_B{}^{(\alpha,\beta)}$, for all $\alpha, \beta \subseteq U$ such that $\alpha \cap \beta = \phi$.

*Proof:*

Suppose that $\delta_A$ and $\delta_B$ be two BSFS over U.

1. Let $x \in \delta_A{}^{(\alpha,\beta)}$, then $\delta_A{}^{P}(x) \geq \alpha$ and $\delta_A{}^{N}(x) \leq \beta$.
Since $\delta_A \leq \delta_B,\ \alpha \leq \delta_A{}^{P}(x) \leq \delta_B{}^{P}(x)$ and $\delta_A{}^{N}(x) \geq \delta_B{}^{N}(x) \geq \beta$ for all $x \in G$.
$\Rightarrow x \in \delta_B{}^{(\alpha,\beta)}$. Hence $\delta_A{}^{(\alpha,\beta)} \subseteq \delta_B{}^{(\alpha,\beta)}$.
2. Let $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$ and $x \in \delta_A{}^{(\alpha_2,\beta_2)}$ . Then $\delta_A{}^{P}(x) \geq \alpha_2$ and $\delta_A{}^{N}(x) \leq \beta_2$.
Since $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$

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$\Rightarrow \delta_A^{\ P}(x) \ge \alpha_1$ and $\delta_A^{\ N}(x) \le \beta_1$

$\Rightarrow x \in \delta_A^{(\alpha_1,\beta_1)}$

$\therefore\ \delta_A^{(\alpha_2,\beta_2)} \subseteq \delta_A^{(\alpha_1,\beta_1)}$.

3. The proof is directly clear.

*Theorem 3.1*

Let $\delta_A$ and $\delta_B$ be two BSFS over U. $A,B \subseteq E$ and $\alpha,\beta \subseteq U$ such that $\alpha \cap \beta = \phi$.

Then

1. $\delta_A^{(\alpha,\beta)} \cup \delta_B^{(\alpha,\beta)} \subseteq (\delta_A \cup \delta_B)^{(\alpha,\beta)}$
2. $\delta_A^{(\alpha,\beta)} \cap \delta_B^{(\alpha,\beta)} \subseteq (\delta_A \cap \delta_B)^{(\alpha,\beta)}$

*Proof:*

1. For all $x \in E$, let $x \in \delta_A^{(\alpha,\beta)} \cup \delta_B^{(\alpha,\beta)}$

$\Rightarrow$ ( $\delta_A^{\ P}(x) \ge \alpha$ and $\delta_A^{\ N}(x) \le \beta$ ) $\vee$ ( $\delta_B^{\ P}(x) \ge \alpha$ and $\delta_B^{\ N}(x) \le \beta$ )

$\Rightarrow (\delta_A^{\ P}(x) \cup \delta_B^{\ P}(x) \ge \alpha)$ or $(\delta_A^{\ N}(x) \cap \delta_B^{\ N}(x) \le \beta)$

$\Rightarrow x \in (\delta_A \cup \delta_B)^{(\alpha,\beta)}$

Hence $\delta_A^{(\alpha,\beta)} \cup \delta_B^{(\alpha,\beta)} \subseteq (\delta_A \cup \delta_B)^{(\alpha,\beta)}$.

2. Similar to the proof of 1.

*Note 4*

Let I be an index set and $\delta_A$ be a family of BSFS-sets over U. Then, for any $\alpha,\beta \subseteq U$ such that $\alpha \cap \beta = \phi$,

(i) $\bigcup_{i\in I}(\delta_{A_i}^{(\alpha,\beta)}) \subseteq (\bigcup_{i\in I}\delta_{A_i})^{(\alpha,\beta)}$

(ii) $\bigcap_{i\in I}(\delta_{A_i}^{(\alpha,\beta)}) = (\bigcap_{i\in I}\delta_{A_i})^{(\alpha,\beta)}$

*Note 5*

Let $\delta_A$ be a BSFS over U and $\left\{\alpha_i / i\in I\right\}$ and $\left\{\beta_j / i\in I\right\}$ be two non-empty family of subsets of U. If $\alpha = \bigcap\left\{\alpha_i / i\in I\right\}$, $\overline{\alpha} = \bigcup\left\{\alpha_i / i\in I\right\}$, $\beta = \bigcap\left\{\beta_j / i\in I\right\}$ and $\overline{\beta} = \bigcup\left\{\beta_j / i\in I\right\}$, then the following assertions hold,

1. $\bigcup_{i\in I}\delta_A^{(\alpha_i,\beta_j)} \subseteq \delta_A^{(\alpha,\overline{\beta})}$
2. $\bigcap_{i\in I}\delta_A^{(\alpha_i,\beta_j)} = \delta_A^{(\overline{\alpha},\underline{\beta})}$

*Theorem 3.2*

Let $\delta_G$ be a BSFS- subgroup over U and $\alpha,\beta \subseteq U$ such that $\alpha \cap \beta = \phi$. Then $\delta_G^{(\alpha,\beta)}$ is also BSFS-subgroup of G whenever it is non empty.

*Proof:*

It is clear that $\delta_G^{(\alpha,\beta)} \neq \phi$.

Suppose that $x,y \in \delta_G^{(\alpha,\beta)}$, then $\delta_G^{\ P}(x) \ge \alpha$, $\delta_G^{\ P}(y) \ge \alpha$ and $\delta_G^{\ N}(x) \le \beta$, $\delta_G^{\ N}(y) \le \beta$.

$$\begin{aligned}
\inf \delta_G^{\ P}(xy^{-1}) &\ge T\left\{\inf \delta_G^{\ P}(x), \inf \delta_G^{\ P}(y^{-1})\right\}\\
&= T\left\{\inf \delta_G^{\ P}(x), \inf \delta_G^{\ P}(y)\right\}\\
&\ge T(\alpha,\alpha) \ge \alpha\\
\sup \delta_G^{\ P}(xy^{-1}) &\ge T\left\{\sup \delta_G^{\ P}(x), \sup \delta_G^{\ P}(y^{-1})\right\}\\
&= T\left\{\sup \delta_G^{\ P}(x), \sup \delta_G^{\ P}(y)\right\}\\
&\ge T(\alpha,\alpha) \ge \alpha
\end{aligned}$$

and

$$\begin{aligned}
\inf \delta_G^{\ N}(xy^{-1}) &\le S\left\{\inf \delta_G^{\ N}(x), \inf \delta_G^{\ N}(y^{-1})\right\}\\
&= S\left\{\inf \delta_G^{\ N}(x), \inf \delta_G^{\ N}(y)\right\}\\
&\le S(\beta,\beta) \le \beta\\
\sup \delta_G^{\ N}(xy^{-1}) &\le S\left\{\sup \delta_G^{\ N}(x), \sup \delta_G^{\ N}(y^{-1})\right\}\\
&= S\left\{\sup \delta_G^{\ N}(x), \sup \delta_G^{\ N}(y)\right\}\\
&\le S(\beta,\beta) \le \beta
\end{aligned}$$

$\therefore\ xy^{-1} \in \delta_G^{(\alpha,\beta)}$ and $\delta_G^{(\alpha,\beta)}$ is a BSFS-subgroup of G.

*Theorem 3.3*

Let $\delta_{G_i}$ be a family of BSFS-subgroup over U for all $i \in I$. Then $\bigcap_{i\in I}\delta_{G_i}$ is a BSFS-subgroup over U.

*Proof:*

Let $x,y \in G$. Since $\delta_{G_i}$ be a BSFS-subgroup over U for all $i \in I$.

This implies that $\delta_{G_i}^{\ P}(x\,y^{-1}) \ge T\left\{\delta_{G_i}^{\ P}(x),\ \delta_{G_i}^{\ P}(y)\right\}$ for all $i \in I$.

Then

$$\begin{aligned}
\inf \delta_{G_i}^{\ P}(x\,y^{-1}) &\ge T\left\{\inf \delta_{G_i}^{\ P}(x), \inf \delta_{G_i}^{\ P}(y)\right\}\\
\inf\left(\bigcap_{i\in I}\delta_{G_i}^{\ P}(x\,y^{-1})\right) &\ge \bigcap_{i\in I} T\left\{\inf \delta_{G_i}^{\ P}(x), \inf \delta_{G_i}^{\ P}(y)\right\}\\
&= T\left\{\inf\left(\bigcap_{i\in I}\delta_{G_i}^{\ P}(x)\right), \inf\left(\bigcap_{i\in I}\delta_{G_i}^{\ P}(y\right.\right.
\end{aligned}$$

and

$$\begin{aligned}
\sup \delta_{G_i}^{\ P}(x\,y^{-1}) &\ge T\left\{\sup \delta_{G_i}^{\ P}(x), \sup \delta_{G_i}^{\ P}(y)\right\}\\
\sup\left(\bigcap_{i\in I}\delta_{G_i}^{\ P}(x\,y^{-1})\right) &\ge \bigcap_{i\in I} T\left\{\sup \delta_{G_i}^{\ P}(x), \sup \delta_{G_i}^{\ P}(y)\right\}\\
&= T\left\{\sup\left(\bigcap_{i\in I}\delta_{G_i}^{\ P}(x)\right), \sup\left(\bigcap_{i\in I}\delta_{G_i}^{\ P}(y\right.\right.\\
&= T\left\{\sup \delta_{G_i}^{\ P}(x), \sup \delta_{G_i}^{\ P}(y)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\inf \delta_{G_i}^{\ N}(x\,y^{-1}) &\le S\left\{\inf \delta_{G_i}^{\ N}(x), \inf \delta_{G_i}^{\ N}(y)\right\}\\
\inf\left(\bigcup_{i\in I}\delta_{G_i}^{\ N}(x\,y^{-1})\right) &\le \bigcup_{i\in I} S\left\{\inf \delta_{G_i}^{\ N}(x), \inf \delta_{G_i}^{\ N}(y)\right\}
\end{aligned}$$

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$$= S\left\{ \inf\left(\bigcup_{i\in I}\delta_{G_i}{}^{N}(x)\right), \inf\left(\bigcup_{i\in I}\delta_{G_i}{}^{N}(y)\right)\right\}$$

and

$$\sup \delta_{G_i}{}^{N}(x y^{-1}) \le S\left\{\sup \delta_{G_i}{}^{N}(x), \sup \delta_{G_i}{}^{N}(y)\right\}$$

$$\sup\left(\bigcup_{i\in I}\delta_{G_i}{}^{N}(x y^{-1})\right) \le \bigcup_{i\in I} S\left\{\sup \delta_{G_i}{}^{N}(x), \sup \delta_{G_i}{}^{N}(y)\right\}$$

$$= S\left\{ \sup\left(\bigcup_{i\in I}\delta_{G_i}{}^{N}(x)\right), \sup\left(\bigcup_{i\in I}\delta_{G_i}{}^{N}(y)\right)\right\}$$

Thus $\bigcap_{i\in I}\delta_{G_i}$ is a BSFS-subgroup over U.

*Theorem 3.4*

Let $\delta_G$ be a BSFS-subgroup over U. Then $\delta_G(x^n) \ge \delta_G(x)$ for all $x \in G$ where $n \in N$.

*Proof:*

Suppose that $\delta_G$ is a BSFS-subgroup over U. Then

$$\delta_G{}^{P}(x^n) \ \ge \delta_G{}^{P}(x) \cap \delta_G{}^{P}(x) \cap \cdots \cap \delta_G{}^{P}(x)$$

( n-times)

$$\therefore\ \inf\left(\delta_G{}^{P}(x^n)\right) \ge T\left\{\inf\left(\delta_G{}^{P}(x)\right), \inf\left(\delta_G{}^{P}(x)\right), \cdots, \inf\left(\delta_G{}^{P}(x)\right)\right\}$$

and

$$\delta_G{}^{N}(x^n) \ \le \delta_G{}^{N}(x) \cup \delta_G{}^{N}(x) \cup \cdots \cup \delta_G{}^{N}(x)$$

( n-times)

$$\therefore\ \inf\left(\delta_G{}^{N}(x^n)\right) \le S\left\{\inf\left(\delta_G{}^{N}(x)\right), \inf\left(\delta_G{}^{N}(x)\right), \cdots, \inf\left(\delta_G{}^{N}(x)\right)\right\}$$

Similarly

$$\sup\left(\delta_G{}^{P}(x^n)\right) \ \ge T\left\{\sup\left(\delta_G{}^{P}(x)\right), \sup\left(\delta_G{}^{P}(x)\right), \cdots, \sup\left(\delta_G{}^{P}(x)\right)\right\}$$

$$\sup\left(\delta_G{}^{N}(x^n)\right) \ \le S\left\{\sup\left(\delta_G{}^{N}(x)\right), \sup\left(\delta_G{}^{N}(x)\right), \cdots, \sup\left(\delta_G{}^{N}(x)\right)\right\}$$

for all $x \in G$.

Thus $\delta_G(x^n) \ge \delta_G(x)$.

*Theorem 3.5*

Let $\delta_G$ be a BSFS-subgroup over U. If for all $x, y \in G$,

$\inf\left(\delta_G{}^{P}(x y^{-1})\right) = U$ and $\inf\left(\delta_G{}^{N}(x y^{-1})\right) = \phi$

$\sup\left(\delta_G{}^{P}(x y^{-1})\right) = U$ and $\sup\left(\delta_G{}^{N}(x y^{-1})\right) = \phi$

Then $\inf \delta_G{}^{P}(x) = \inf \delta_G{}^{P}(y)$ and $\sup \delta_G{}^{N}(x) = \sup \delta_G{}^{N}(y)$.

*Proof:*

For any $x, y \in G$

$$\begin{aligned}\inf\left(\delta_G{}^{P}(x)\right) &= \inf\left(\delta_G{}^{P}(x y^{-1})y\right)\\ &\ge T\left\{\inf \delta_G{}^{P}(x y^{-1}), \inf \delta_G{}^{P}(y)\right\}\\ &= T\left\{U, \inf \delta_G{}^{P}(y)\right\}\\ &= \inf \delta_G{}^{P}(y)\end{aligned}$$

and

$$\begin{aligned}\inf\left(\delta_G{}^{P}(y)\right) &= \inf\left(\delta_G{}^{P}(y^{-1})\right)\\ &= \inf\left(\delta_G{}^{P}(x^{-1}(x y^{-1}))\right)\\ &\ge T\left\{\inf \delta_G{}^{P}(x^{-1}), \inf \delta_G{}^{P}(x y^{-1})\right\}\\ &= T\left\{\inf \delta_G{}^{P}(x^{-1}), U\right\}\\ &= \inf \delta_G{}^{P}(x)\end{aligned}$$

Thus $\inf \delta_G{}^{P}(x) = \inf \delta_G{}^{P}(y)$.

Also

$$\begin{aligned}\sup\left(\delta_G{}^{N}(x)\right) &= \sup\left(\delta_G{}^{N}(x y^{-1})y\right)\\ &\le S\left\{\sup \delta_G{}^{N}(x y^{-1}), \sup \delta_G{}^{N}(y)\right\}\\ &= S\left\{\phi, \sup \delta_G{}^{N}(y)\right\}\\ &= \sup \delta_G{}^{N}(y)\end{aligned}$$

and

$$\begin{aligned}\sup\left(\delta_G{}^{N}(y)\right) &= \sup\left(\delta_G{}^{N}(y^{-1})\right)\\ &= \sup\left(\delta_G{}^{N}(x^{-1}(x y^{-1}))\right)\\ &\le S\left\{\sup \delta_G{}^{N}(x^{-1}), \sup \delta_G{}^{N}(x y^{-1})\right\}\\ &= S\left\{\sup \delta_G{}^{N}(x^{-1}), \phi\right\}\\ &= \sup \delta_G{}^{N}(x)\end{aligned}$$

Thus $\sup \delta_G{}^{N}(x) = \sup \delta_G{}^{N}(y)$.

## 4. Conclusion:

Bipolar fuzzy soft set and bipolar fuzzy soft star shaped set are some special fuzzy sets. In this paper, we introduce some new different types of bipolar fuzzy soft star shaped set. By discussing the relationship among these different types of star shapedness and obtained some different properties.

*Future work*

One can obtain the similar concept in the field of rough set and vague set.

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